FluidMaths

Year 2 (A-Level)

Differentiation Set 1B (Solutions)
(Product and Quotient Rules, Implicit differentiation and Rates of change)

The marks shown are for guidance purposes only

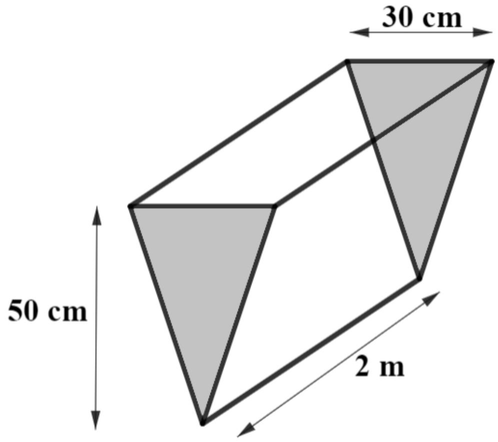
When not specified, round all non-terminating decimals to 3 significant figures

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$	$\dfrac{\mathrm{f}'(x)\mathrm{g}(x) - \mathrm{f}(x)\mathrm{g}'(x)}{(\mathrm{g}(x))^2}$

1	A curve is defined implicitly by the equation $$x^2+y^2-kxy+\frac{1}{2}kx-2y+5=0$$ The gradient of the curve at $(-1,-1)$ is 5 Calculate the value of integer k **Solution** $x^2+y^2-kxy+\frac{1}{2}kx-2y+5=0$ Implicit differentiation $2x+2y\frac{dy}{dx}-kx\frac{dy}{dx}-ky+\frac{1}{2}k-2\frac{dy}{dx}=0$ **[3marks]** $\frac{dy}{dx}(2y-kx-2)=-2x+ky-\frac{1}{2}k$ $\frac{dy}{dx}=\frac{-2x+ky-\frac{1}{2}k}{(2y-kx-2)}$ **[1mark]** Substitute $(-1,-1)$ and $\frac{dy}{dx}=5$ into $\frac{-2x+ky-\frac{1}{2}k}{(2y-kx-2)}$ $5=\frac{-2(-1)+k(-1)-\frac{1}{2}k}{2(-1)-k(-1)-2}$ **[1mark]** $5=\frac{2-\frac{3}{2}k}{k-4}$ $5k-20=2-\frac{3}{2}k$ $5k+\frac{3}{2}k=22$ $\frac{13}{2}k=22$ $k=\frac{44}{13}$ **[1mark]**

2 Here is a water trough of length 2 m



The perpendicular height of the trough is 50 cm
The rectangular top of the trough is 30 cm wide
The trough is filled with water to the brim and some lambs are drinking steadily from it.
When the water level was 15 cm from the bottom of the trough, it was being lowered at a rate of 2.5 cm/min
Calculate the amount of water the lambs are drinking per minute.

Solution

The rate of change of the volume of water in the trough per unit time = The volume of water consumed by lambs per unit minute

Let the amount of water the lambs are drinking per minute be α

Then, $\frac{dV}{dt} = -\alpha$

Negative shows that amount of water is decreasing

$\frac{dV}{dt} = -\alpha$ **[1mark]**

$\frac{d(xyz)}{dt} = -\alpha$ **[1mark]**

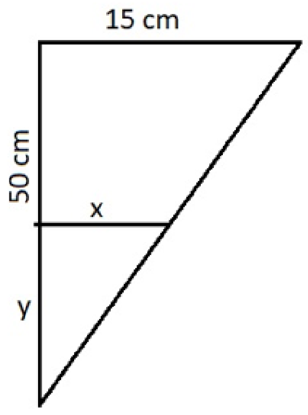
$z = 2\text{m} = 200\text{cm}$

$y =$ height at any time

$x =$ width at any time

$\frac{y}{x} = \frac{50}{15}$

$x = \frac{15}{50} y$



$\frac{d(xyz)}{dt} = -\alpha$

$\frac{d[\frac{15}{50} y(y)(200)]}{dt} = -\alpha$ **[1mark]**

$60\frac{d(y^2)}{dt} = -\alpha$

$120y\frac{dy}{dt} = -\alpha$

Substitute $y = 15$ and $\frac{dy}{dt} = -2.5$

$120(15)(-2.5) = -\alpha$ **[1mark]**

$\alpha = 4500\ \text{cm}^3/\text{min}$ **[1mark]**

3	A function is defined on a set of real numbers as $f(x) = 3x^2 \ln 2x$ a) Find the roots of the equation $3x^2 \ln 2x = 0$ b) Determine the stationary points of f(x) Give your answers to 3 significant figures c) Determine the exact equation of a tangent to f(x) at $x = 1$ **Solution** a) $3x^2 \ln 2x = 0$ Either $x = 0$ or $\ln 2x = 0$ $2x = e^0$ $2x = 1$ Therefore, the roots are $x = 0$ or $x = \frac{1}{2}$ **[2marks]** b) $f(x) = 3x^2 \ln 2x$ Apply the product rule to find $f'(x)$ $f'(x) = 6x \ln 2x + 3x^2 \left(\frac{1}{2x}\right)$ **[2marks]** $f'(x) = 6x \ln 2x + \frac{3}{2}x$ $f'(x) = 3x\left(2 \ln 2x + \frac{1}{2}\right)$ **[1mark]** At stationary points $f'(x) = 0$ $3x\left(2 \ln 2x + \frac{1}{2}\right) = 0$ **[1mark]** Either $3x = 0$ therefore $x = 0$ or $2 \ln 2x + \frac{1}{2} = 0$ $\ln 2x = -\frac{1}{4}$ $2x = e^{-\frac{1}{4}}$

$x = \frac{1}{2}e^{-\frac{1}{4}} = 0.389$ **[2marks]**

when $x = 0.389$

$\text{f}(0.389) = 3(0.389)^2 \ln\big(2(0.389)\big)$

$= -0.114$ **[1mark]**

Hence, the stationary point is $(0.389, -0.114)$

c)

$\text{f}(x) = 3x^2 \ln 2x$

$\text{f}(1) = 3 \ln 2$ **[1mark]**

$\text{f}'(1) = 3\left(2 \ln 2 + \frac{1}{2}\right)$ **[1mark]**

Hence, the gradient of the tangent is $6 \ln 2 + \frac{3}{2}$

So tangent equation to f(*x*) at $x = 1$ is

$y - 3 \ln 2 = 6 \ln 2 + \frac{3}{2}\ (x - 1)$ **[1mark]**

$y - 3 \ln 2 = \left(6 \ln 2 + \frac{3}{2}\right)x - 6 \ln 2 - \frac{3}{2}$

$y = \left(6 \ln 2 + \frac{3}{2}\right)x - 6 \ln 2 + 3 \ln 2 - \frac{3}{2}$

$y = \left(6 \ln 2 + \frac{3}{2}\right)x - \left(3 \ln 2 + \frac{3}{2}\right)$ **[2marks]**

4	A curve is defined by the equation $\frac{3x^2-y^2}{2x}=5$ Determine the equation of a normal to the curve at $x=4$ Give your answer in the form $\mathrm{A}y+\mathrm{B}x+\mathrm{C}=0$ Where A, B and C are exact constants **Solution** $\frac{3x^2-y^2}{2x}=5$ $3x^2-y^2=10x$ Substitute $x=4$ into $3x^2-y^2=10x$ $3(16)-y^2=10(4)$ **[1mark]** $48-y^2=40$ $y^2=8$ $y=\sqrt{8}=2\sqrt{2}$ **[1mark]** Differentiate $3x^2-y^2=10x$ implicitly $6x-2y\frac{dy}{dx}=10$ **[2marks]** Substitute $x=4$ and $y=2\sqrt{2}$ $6(4)-2(2\sqrt{2})\frac{dy}{dx}=10$ **[1mark]** $24-4\sqrt{2}\frac{dy}{dx}=10$ $4\sqrt{2}\frac{dy}{dx}=-14$ $\frac{dy}{dx}=\frac{14}{4\sqrt{2}}=\frac{7}{2\sqrt{2}}$ {Rationalise} Hence, $\frac{dy}{dx}=\frac{7\sqrt{2}}{4}$ **[1mark]**

Therefore, the gradient of the normal is $\frac{7\sqrt{2}}{4} = -\frac{4}{7\sqrt{2}}$

Rationalize to remove the radical denominator

$-\frac{4}{7\sqrt{2}} = -\frac{2\sqrt{2}}{7}$ **[1mark]**

The equation of the normal is

$y - 2\sqrt{2} = -\frac{2\sqrt{2}}{7}(x - 4)$ **[1mark]**

$y - 2\sqrt{2} = -\frac{2\sqrt{2}}{7}x + \frac{8\sqrt{2}}{7}$

$7y - 14\sqrt{2} = -2\sqrt{2}x + 8\sqrt{2}$

$7y + (2\sqrt{2})x - 22\sqrt{2} = 0$ **[1mark]**

5 A function is defined on a set of real numbers as

$$\mathrm{f}(x) = \frac{kx^2-5}{3x-2}; x \neq \frac{2}{3}$$

a) Given that $\mathrm{f}'(-1) = \frac{29}{25}$ find the value of k.

b) Hence, Evaluate $\mathrm{f}''(-1)$

Solution

a) $\mathrm{f}(x) = \frac{kx^2-5}{3x-2}$

apply the quotient rule

$\mathrm{f}'(x) = \frac{(3x-2)2kx-3(kx^2-5)}{(3x-2)^2}$ **[2marks]**

$\mathrm{f}'(-1) = \frac{(-5)(-2k)-3(k-5)}{25}$ **[1mark]**

$\frac{29}{25} = \frac{(-5)(-2k)-3(k-5)}{25}$ **[1mark]**

$29 = 10k - 3k + 15$

$7k = 14$

Hence, $k = 2$ **[1mark]**

b)

$f'(x) = \frac{3kx^2-4kx+15}{(3x-2)^2}$

Substitute $k = 2$ into $f'(x)$

$f'(x) = \frac{3(2)x^2-4(2)x+15}{(3x-2)^2}$

Hence, $f'(x) = \frac{6x^2-8x+15}{(3x-2)^2}$ **[1mark]**

$f''(x) = \frac{(3x-2)^2(12x-8)-6(3x-2)(6x^2-8x+15)}{(3x-2)^4}$ **[2marks]**

Therefore, $f''(-1) = \frac{25(-20)-6(-5)(29)}{(-5)^4}$ **[1mark]**

$f''(-1) = \frac{25(-20)-6(-5)(29)}{(-5)^4}$

Hence, $f''(-1) = -\frac{370}{625}$ **[1mark]**

6	The function is defined by the equation $$f(x) = \sin x + px\cos x$$ When $x = \frac{\pi}{6}, y = \frac{\pi\sqrt{3}+3}{6}$ a) Find the value of the constant p b) Hence, evaluate $f''\left(\frac{\pi}{2}\right)$ **Solution** a) $f(x) = \sin x + px\cos x$ $f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) + p\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$ **[1mark]** $\frac{\pi\sqrt{3}}{6} + \frac{1}{2} = \frac{1}{2} + p\left(\frac{\pi}{6}\right)\frac{\sqrt{3}}{2}$ **[1mark]** $\frac{\pi\sqrt{3}}{6} = p\left(\frac{\pi}{6}\right)\frac{\sqrt{3}}{2}$ $\frac{\pi\sqrt{3}}{6} = \frac{p\pi\sqrt{3}}{12}$ Hence, $p = 2$ **[1mark]** b) $f(x) = \sin x + 2x\cos x$ $f'(x) = \cos x + 2\cos x - 2xsinx$ **[2marks]** $f'(x) = 3\cos x - 2xsinx$ $f''(x) = -3\sin x - 2sinx - 2xcosx$ **[2marks]** $f''(x) = -5sinx - 2xcosx$ Hence, $f''\left(\frac{\pi}{2}\right) = -5$ **[1mark]**