FluidMaths

Year 2 (A-Level)

Differentiation Set 2A - (Solutions)
(Chain Rule, e^x, ln$x$ and Trig Functions)

The marks shown are for guidance purposes only

The questions are repeated here for your convenience

1	A curve is defined by the equation $y = 5e^{x^3}$ Show that the equation of a tangent to the curve at $x = \frac{1}{2}$ can be written as $y = \frac{5}{8}e^{\frac{1}{8}}(6x + 5)$ **Solution** $y = 5e^{x^3}$ Therefore, $\frac{dy}{dx} = 3x^2 5e^{x^3}$ **[2marks]** When $x = \frac{1}{2}$ $y = 5e^{\frac{1}{8}}$ **[1mark]** Therefore, $\left.\frac{dy}{dx}\right\|_{x=\frac{1}{2}} = \frac{15}{4}e^{\frac{1}{8}}$ **[1mark** The equation of the tangent is $\left(y - 5e^{\frac{1}{8}}\right) = \frac{15}{4}e^{\frac{1}{8}}\left(x - \frac{1}{2}\right)$ **[1mark]** $y = 5e^{\frac{1}{8}} + \frac{15}{4}e^{\frac{1}{8}}\left(x - \frac{1}{2}\right)$ $y = 5e^{\frac{1}{8}} + \frac{15}{4}xe^{\frac{1}{8}} - \frac{15}{8}e^{\frac{1}{8}}$ {Rearrange} $y = \frac{15}{4}xe^{\frac{1}{8}} + \frac{25}{8}e^{\frac{1}{8}}$ {Factorise} Hence, $y = \frac{5}{8}e^{\frac{1}{8}}(6x + 5)$ **[2marks]**

2	A curve is defined by the equation $y = \sqrt{kx^2 + 4}$ $y = -\frac{3}{2}x + 1$ is a tangent to the curve at $x = -2$ a) Calculate the value of the whole number k b) Find the equation of a normal to the curve at $x = 4$ c) Hence, determine the point where the normal and the tangent intersect. **Solution** a) When $x = -2$ Then, $y = -\frac{3}{2} \times -2 + 1 = 4$ **[1mark]** Therefore, the point $(-2,4)$ is on the curve So, $4 = \sqrt{k(-2)^2 + 4}$ **[1mark]** $4 = \sqrt{4k + 4}$ {Square both sides} $16 = 4k + 4$ $4k = 12$ $k = 3$ **[1mark]** Hence, $y = \sqrt{3x^2 + 4}$

b) Let $u = 3x^2 + 4$ then $y = u^{\frac{1}{2}}$

$\frac{du}{dx} = 6x$ and $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$

Therefore, $\frac{dy}{dx} = \frac{3x}{\sqrt{3x^2+4}}$ **[2marks]**

When $x = 8$; $\left.\frac{dy}{dx}\right|_{x=8} = \frac{3\times 8}{\sqrt{3(8)^2+4}} = \frac{12}{7}$

Therefore, the gradient of the normal is $-\frac{7}{12}$ **[1mark]**

When $x = 8$ then, $y = \sqrt{3(8)^2 + 4} = 14$ **[1mark]**

Therefore, the equation of the normal is

$y - 14 = -\frac{7}{12}(x - 8)$ **[1mark]**

$y - 14 = -\frac{7}{12}x + \frac{14}{3}$

$y = -\frac{7}{12}x + \frac{56}{3}$ **[1mark]**

c) $-\frac{3}{2}x + 1 = -\frac{7}{12}x + \frac{56}{3}$

$-\frac{3}{2}x + \frac{7}{12}x = \frac{56}{3} - 1$

$-\frac{11}{12}x = \frac{53}{3}$

$-33x = 636$ **[1mark]**

$x = -\frac{212}{11}$

$y = -\frac{3}{2}\left(-\frac{212}{11}\right) + 1 = \frac{329}{11}$ **[1mark]**

Hence, the tangent and the normal intersect at $\left(-\frac{212}{11}, \frac{329}{11}\right)$

3 A curve is defined by the equation $y = 5e^{\cos(5x)}$

a) Given that (π, k) is a coordinate on the curve.
Find the exact value of the real number k.

b) Find $\frac{dy}{dx}$

c) Hence, evaluate $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$
Give your answer as an exact value

Solution

a) $y = 5e^{\cos(5\pi)} = 5e^{-1}$ **[1mark]**

Hence, the exact value of $k = \frac{5}{e}$ **[1mark]**

b) Let $u = \cos(5x)$ then $y = 5e^{u}$

$\frac{du}{dx} = -5\sin(5x)$ and $\frac{dy}{du} = 5e^{u}$ **[2marks]**

Therefore, $\frac{dy}{dx} = 5e^{\cos(5x)}(-5\sin 5x)$ **[1mark]**

Hence, $\frac{dy}{dx} = -25\sin(5x)\, e^{\cos(5x)}$

c) $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = -25\sin\left(\frac{5\pi}{2}\right) e^{\cos\left(\frac{5\pi}{2}\right)}$ **[1mark]**

$= -25 \times 1 \times e^{0}$

Hence, $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = -25$ **[1mark]**

4	The equation of a curve is given as $y = 8^{x-2}$ a) The point $A = (k, 10)$ lies on the curve. Determine the exact value of k b) Find the gradient function of the curve **Solution** a) Substitute $y = 10$ into $y = 8^{x-2}$ $10 = 8^{k-2}$ $\ln 10 = (k-2)\ln 8$ **[1mark]** $\frac{\ln 10}{\ln 8} = k - 2$ $k = \frac{\ln 10}{\ln 8} + 2$ **[1mark]** Hence, $k = \frac{\ln 10 + 2\ln 8}{\ln 8}$ b) Apply the exponent law $\{a^b = e^{b\ln a}\}$ Therefore, $y = 8^{x-2}$ should be written as $y = e^{(x-2)\ln 8}$ Now use the chain rule to find $\frac{dy}{dx}$ Let $u = (x-2)\ln 8$ then $y = e^u$ Note that $8 = 2^3$ Therefore, $u = (x-2)\ln 8$ can be written as $3\ln 2\,(x-2)$ You can simplify further by expanding the brackets here So $u = 3\ln 2\, x - 6\ln 2$ Therefore, $\frac{du}{dx} = 3\ln 2$ and $\frac{dy}{du} = e^u$ **[2marks]** $\frac{dy}{dx} = 3\ln 2\, e^{(x-2)\ln 8}$ {Notice that $e^{\ln 8} = 8$} Hence, $\frac{dy}{dx} = 3\ln 2\, 8^{(x-2)}$ **[1mark]**

5	A curve is defined by the equation $y = (8x-3)^n$ Where n is a whole number. a) Given that $\frac{dy}{dx}$ at $x = \frac{1}{2}$ is 96 find the possible values of n b) Find the two possible expressions for $\frac{d^2y}{dx^2}$ **Solution** a) $y = (8x-3)^n$ $\frac{dy}{dx} = 8n(8x-3)^{n-1}$ **[1mark]** Substitute, $x = \frac{1}{2}$ and $\frac{dy}{dx} = 96$ $96 = 8n(n-1)\left(8\left(\frac{1}{2}\right)-3\right)^{n-1}$ **[1mark]** $96 = 8n(n-1)(1)^{n-1}$ Notice that $(1)^{n-1}$ is always equal to 1 **[1mark]** Therefore, $96 = 8n(n-1)$ $n^2 - n = 12$ Therefore, $n^2 - n - 12 = 0$ **[1mark]** $(n-4)(n+3) = 0$ Hence, $n = 4$ or $n = -3$ **[1mark]** b) So, $y = (8x-3)^4$ or $y = (8x-3)^{-3}$ Therefore, $\frac{dy}{dx} = 32(8x-3)^3$ or $\frac{dy}{dx} = -24(8x-3)^{-4}$ **[2marks]** Hence, $\frac{d^2y}{dx^2} = 768(8x-3)^2$ or $\frac{d^2y}{dx^2} = 768(8x-3)^{-5}$ **[2marks]**

6 The function $f(x) = 5^{x+1} - 5^x - 1$ has one real root

a) Find the exact value of this root

b) Determine the equation of a tangent to $f(x)$ at $x = 1$

Solution

a) $5^{x+1} - 5^x - 1 = 0$

$5 \times 5^x - 5^x - 1 = 0$ **[1mark]**

Let $y = 5^x$

Then, we have $5y - y = 1$

$y = \frac{1}{4}$ **[1mark]**

Therefore, $5^x = \frac{1}{4}$

$x \ln 5 = \ln\left(\frac{1}{4}\right)$

Hence, $x = \frac{\ln\left(\frac{1}{4}\right)}{\ln 5}$ **[1mark]**

b) $f(x) = 5^{x+1} - 5^x - 1$

$f(x) = 5 \times 5^x - 5^x - 1$ {Simplify the like terms}

$f(x) = 4 \times 5^x - 1$ **[1mark]**

Therefore, $f'(x) = 4 \ln 5 \, (5^x)$ **[1mark]**

When $x = 1$ we have $f'(1) = 4 \ln 5 \, (5^1)$

$f'(1) = 20 \ln 5$ **[1mark]**

Therefore, the gradient of the tangent is $20 \ln 5$

When $x = 1$ we have $f(1) = 5^{1+1} - 5^1 - 1 = 19$

[1mark]

Hence, the equation of the tangent is

$y - 19 = 20 \ln 5(x - 1)$ **[1mark]**

$y = (20 \ln 5)x - 20 \ln 5 + 19$ **[1mark]**