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Year 2 (A-Level)

Parametric Equations – Set 1
(Solutions)

The marks shown are for guidance purposes only

The questions are repeated here for your convenience

1 A parametric curve is defined by the parameters $x = t^2$ and $y = 2t + 3$

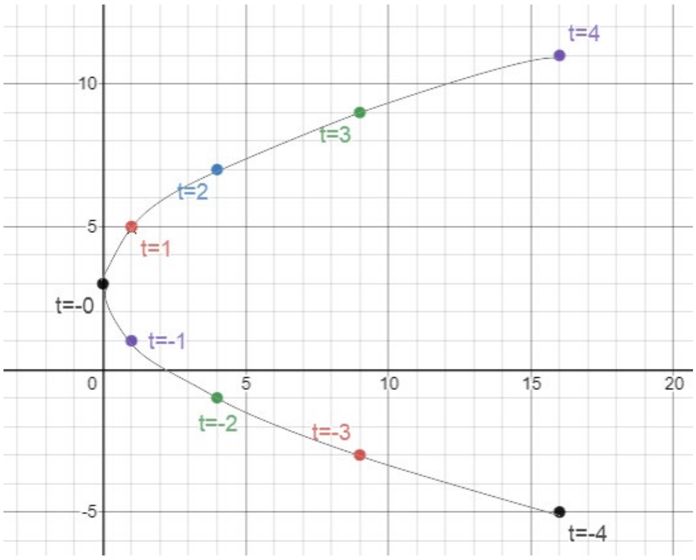
a) Sketch a graph of the curve for $-4 \leq t \leq 4$

b) Find an equation that connects y to x explicitly

<u>Solution</u>

a) Use a table as shown below

t	x	y
-4	16	-5
-3	9	-3
-2	4	-1
-1	1	1
0	0	3
1	1	5
2	4	7
3	9	9
4	16	11



[2marks]

b) Using equation (1) we can find, $t = \pm\sqrt{x}$

Therefore, equation (2) can be written as:

$y = 2(\sqrt{x}) + 3 = 2\sqrt{x} + 3$ and

$y = 2(-\sqrt{x}) + 3 = -2\sqrt{x} + 3$

$\boxed{y = 2\sqrt{x} + 3}$ $\boxed{y = -2\sqrt{x} + 3}$ **[2marks]**

2	Given that $x = 1 - t$ and $y = \frac{t}{t^2-2}$ a) Find an equation in terms of *x* and *y* only b) Hence find $\frac{dy}{dx}$

Solution

a) $t = 1 - x$

Therefore,

$$y = \frac{1-x}{(1-x)^2 - 2} = \frac{1-x}{x^2 - 2x - 1}$$

Therefore, $\boxed{y = \frac{1-x}{x^2-2x-1}}$ **[2marks]**

b) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{t}{t^2-2}\right)$

Using quotient rule:

$$\frac{dy}{dt} = \frac{\frac{d}{dt}(t)(t^2-2) - \frac{d}{dt}(t^2-2)(t)}{(t^2-2)^2}$$

$$\frac{dy}{dt} = \frac{(1)(t^2-2)-(2t)(t)}{(t^2-2)^2}$$

$$\frac{dy}{dt} = \frac{-t^2-2}{(t^2-2)^2}$$

[3marks]

Similarly,

$\frac{dx}{dt} = \frac{d}{dt}(1-t) = -1$ **[1mark]**

Therefore

$\frac{dy}{dx} = \frac{-t^2-2}{(t^2-2)^2} \times -1$ **[1mark]**

$\boxed{\frac{dy}{dx} = \frac{t^2+2}{t^4-4t^2+4}}$ **[1mark]**

Or in the Cartesian form: $\frac{dy}{dx} = \frac{x^2-2x+3}{(x^2-2x-1)^2}$

3 Given that $x = \frac{2t}{t+2}$ and $y = \frac{3t}{t+3}$

a) Show that $6(y - x) = xy$

b) Hence find the equation of a tangent to the curve at $(-3, -2)$

Solution

a) $x(t + 2) = 2t$

$xt + 2x = 2t$

$2x = 2t - xt$

$2x = t(2 - x)$

$t = \frac{-2x}{x-2}$ [**1mark**]

Similarly

$y(t + 3) = 3t$

$yt + 3y = 3t$

$3y = 3t - yt$

$3y = t(3 - y)$

$t = \frac{-3y}{y-3}$ **[1mark]**

Therefore

$\frac{-2x}{x-2} = \frac{-3y}{y-3}$

$-2x(y - 3) = -3y(x - 2)$

$2xy - 6x = 3xy - 6y$

$-xy = 6x - 6y$

$6y - 6x = xy$

$\boxed{6(y - x) = xy}$ **[2marks]**

b) To find the gradient of the tangent, we need to differentiate
Use implicit differentiation:

$6(y - x) = xy$

$6y - 6x = xy$

$\frac{dy}{dx} 6y - \frac{d}{dx}(6x) = \frac{d}{dx}(xy)$

$6\frac{dy}{dx} - 6 = y + x\frac{dy}{dx}$

$6\frac{dy}{dx} - x\frac{dy}{dx} = y + 6$

$\frac{dy}{dx} = \frac{6+y}{6-x}$ **[3marks]**

Now sub the coordinate into $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{6-2}{6-(-3)} = \frac{4}{9}$ **[1mark]**

Therefore, the equation of the tangent will be $y = \frac{4}{9}x + c$

Find c

$-2 = \frac{4}{9} \times (-3) + c$

$c = -\frac{6}{9} = -\frac{2}{3}$

Hence, the equation of the tangent will be, $y = \frac{4}{9}x - \frac{2}{3}$ **[1mark]**

4	Two curves P and Q are defined parametrically as $P:\begin{cases} x = 2t+1 \\ y = -3t^2+5 \end{cases}$ and $Q:\begin{cases} x = 4t+1 \\ y = 5t^2-1 \end{cases}$ a) Determine the coordinates at which the two curves intersect. b) Hence find the size of the area bound between them **<u>Solution</u>** a) Determine their cartesian equations first For P $x-1=2t$ $t=\frac{x-1}{2}$, Therefore, $y=-3\left(\frac{x-1}{2}\right)^2+5$ $y=-\frac{3}{4}(x-1)^2+5$ **[2marks]** For Q $x-1=4t$ $t=\frac{x-1}{4}$ So, $y=5\left(\frac{x-1}{4}\right)^2-1$ $y=\frac{5}{16}(x-1)^2-1$ **[2marks]** Therefore, $-\frac{3}{4}(x-1)^2+5=\frac{5}{16}(x-1)^2-1$ **[1mark]** $\frac{17}{16}(x-1)^2=6$ $(x-1)^2=\frac{96}{17}$ **[1mark]** $x=3.38(3\text{sf})$ and $x=-1.38(3\text{sf})$ **[1mark]** The corresponding y-values are: $y=-\frac{3}{4}(3.38-1)^2+5=0.752\ (3\text{sf})$ **[1mark]** $y=-\frac{3}{4}(-1.38-1)^2+5=0.752\ (3\text{sf})$ **[1mark]** Hence, the points of intersection are $\boxed{(3.38, 0.752);\ (-1.38, 0.752)}$

b) The area between the curves;

$$\int_a^b |f(x) - g(x)| dx$$

$$= \int_{-1.38}^{3.38} \left| \left(-\frac{3}{4}(x-1)^2 + 5 \right) - \left(\frac{5}{16}(x-1)^2 - 1 \right) \right| dx$$

$$= \int_{-1.38}^{3.38} \left| \left(-\frac{17}{16}(x-1)^2 + 6 \right) \right| dx$$

$$= \int_{-1.38}^{3.38} \left| \left(-\frac{17}{16}(x-1)^2 \right) \right| dx + \int_{-1.38}^{3.38} |6| dx$$

$= -9.50 + 28.5 = 19.0$(3sf)

Hence, The area between the two curves $= 19.0$ (3sf) **[5marks]**

5 A curve is defined in terms of the parameter t as

$x = 2\sin t + 10$ and $y = -2\cos t - 1$

a) Show that the curve is a circle and hence state its radius and centre

b) The line $y = -\frac{1}{3}x + 1$ crosses the curve at points A and B.
Determine the coordinates of A and B

Solution

a)

$x - 10 = 2\sin t$ **[1mark]**

$-y - 1 = 2\cos t$ **[1mark]**

Squaring and adding both the equations:

$(x-10)^2 + (-y-1)^2 = 4\sin^2 t + 4\cos^2 t$

$(x-10)^2 + (-y-1)^2 = 4(\sin^2 t + \cos^2 t)$

$(x-10)^2 + (y+1)^2 = 2^2$ **[2marks]**

$\boxed{\text{A circle of Center } (10, -1) \text{ and radius } = 2}$

b) Sub $y = -\frac{1}{3}x + 1$ into the equation of the circle

$(x-10)^2 + \left(\left(-\frac{1}{3}x + 1\right) + 1\right)^2 = 2^2$

$x^2 - 20x + 100 + \left(\left(\frac{1}{3}x - 1\right) + 1\right)^2 = 4$

$x^2 - 20x + 100 + \left(\frac{1}{3}x - 2\right)^2 + 2\left(\frac{1}{3}x - 2\right) + 1 = 4$

$x^2 - 20x + 100 + \frac{1}{9}x^2 - \frac{4}{3}x + 4 + \frac{2}{3}x - 4 = 4$

$10x^2 - 186x + 864 = 0$ Hence, $x = 18$ or $x = 19.2$ **[4marks]**

If $x = 18$ then $y = -\frac{1}{3}(18) + 1 = -5$ **[1mark]**

If $x = 19.2$ then $y = -\frac{1}{3}(19.2) + 1 = -5.4$ **[1mark]**

Hence, the coordinates of A and B are $(18, -5)$ and $(19.2, -5.4)$ respectively

6 A parametric curve is defined by the equations $x = \sin^2 t$ and $y = \sec t$

a) Show that the Cartesian equation for the curve can be written as $y = \frac{1}{\sqrt{1-x}}$

b) Find $\frac{dy}{dx}$

c) Hence find the equation of a normal to the curve at $x = -1$.
Give your answer in the form $Ax + By + C = 0$
where A, B and C are exact values to be found

d) Hence evaluate $\int_{-1}^{0.5} \left(\frac{1}{\sqrt{1-x}}\right) dx$

<u>Solution</u>

a) $y = \frac{1}{cost}$

Using the identity $\sin^2 t + \cos^2 t = 1$ therefore, $\cos^2 t = 1 - \sin^2 t$

Therefore, $\cos t = \sqrt{1 - \sin^2 t}$

Now sub $x = \sin^2 t$

So, $\cos t = \sqrt{1 - x}$

Therefore, $y = \frac{1}{\sqrt{1-x}}$ As required **[2marks]**

b) Apply the chain rule

Let $u = 1 - x$ so $\frac{du}{dx} = -1$ **[1mark]**

Then $y = u^{-\frac{1}{2}}$ so $\frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$ **[1mark]**

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \times (-1)$

$\frac{dy}{dx} = \frac{1}{2u^{\frac{3}{2}}}$

Sub $u = 1 - x$

Therefore, $\frac{dy}{dx} = \frac{1}{2(1-x)^{\frac{3}{2}}}$ **[2marks]**

c) $\left.\frac{dy}{dx}\right|_{x=-1} = \frac{1}{2(1-(-1))^{\frac{3}{2}}} = \frac{1}{2\sqrt{(2)}^{3}} = \frac{1}{2(\sqrt{8})} = \frac{1}{4\sqrt{2}}$ **[2marks]**

Therefore, the gradient of the normal at $x = -1$ will be $-4\sqrt{2}$

So the equation will be $y = -4\sqrt{2}x + c$

At $x = -1$ we can find that, $y = \frac{1}{\sqrt{(1-(-1))}} = \frac{1}{\sqrt{2}}$ **[2marks]**

Hence the point of intersection between the normal and the curve is $\left(-1, \frac{1}{\sqrt{2}}\right)$

Therefore, $\frac{1}{\sqrt{2}} = -4\sqrt{2} \times -1 + c$

$\frac{1}{\sqrt{2}} = 4\sqrt{2} + c$

$c = \frac{1}{\sqrt{2}} - 4\sqrt{2}$

$c = -\frac{7\sqrt{2}}{2}$ **[2marks]**

Therefore, $y = -4\sqrt{2}x - \frac{7\sqrt{2}}{2}$ Hence we have, $8\sqrt{2}x + 2y + 7\sqrt{2} = 0$

Where $A = 8\sqrt{2},\ B = 2$ and $C = 7\sqrt{2}$ **[2marks]**

d)

$$\int_{-1}^{0.5} y\,dx = \int_{-1}^{0.5} \frac{1}{\sqrt{1-x}}\,dx$$

Apply u-substitution:

$$u = 1 - x$$
$$du = -dx$$

$$= -\int_{0.5}^{2} -\frac{1}{\sqrt{u}}\,du = \int_{0.5}^{2} \frac{1}{\sqrt{u}}\,du$$

$$= 2\left[\sqrt{u}\right]_{0.5}^{2} = 2(\sqrt{2} - \sqrt{0.5})$$

Hence, $\boxed{\int_{-1}^{0.5}\left(\frac{1}{\sqrt{1-x}}\right)dx = 1.41}$ (3sf)

[4marks]