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Year 2 (A-Level)

Integration – Set 1B
(Integration by Substitution, by Parts and Diff Equations)
Solutions

The marks shown are for guidance purposes only

The questions have been repeated here for your convenience

1	Given that $u = 1 - e^{3x}$

a)Find $\int \frac{e^{3x}}{\sqrt{1-e^{3x}}} dx$

b) Hence show that $\int_0^2 \frac{e^{3x}}{\sqrt{1-e^{3x}}} dx = -\frac{2}{3}\sqrt{1-e^6}$

Solution

a)$u = 1 - e^{3x}$ Therefore, $e^{3x} = 1 - u$

$\frac{du}{dx} = -3e^{3x}$

$e^{3x}dx = -\frac{1}{3}du$ **[1mark]**

Therefore, $\int \frac{e^{3x}}{\sqrt{1-e^{3x}}} dx = \int \frac{-1}{3\sqrt{u}} du$

$= -\frac{1}{3}\int u^{-\frac{1}{2}} du$ **[1mark]**

$= -\frac{1}{3} \times \frac{u^{\frac{1}{2}}}{\frac{1}{2}}$ **[1mark]**

$= -\frac{1}{3} \times \frac{(1-e^{3x})^{\frac{1}{2}}}{\frac{1}{2}} = \boxed{-\frac{2}{3} \times (1 - e^{3x})^{\frac{1}{2}} + c}$

or $-\frac{2}{3}\sqrt{1 - e^{3x}} + C$ **[2marks]**

b) From **part a**

$$\int \frac{e^{3x}}{\sqrt{1-e^{3x}}}dx = -\frac{2}{3}\times(1-e^{3x})^{\frac{1}{2}} + C$$

Therefore, $\int_0^2 \frac{e^{3x}}{\sqrt{1-e^{3x}}}dx = \left[-\frac{2}{3}\times(1-e^{3x})^{\frac{1}{2}}\right]_0^2$

$= \left(-\frac{2}{3}\times\left(1-e^{3(2)}\right)^{\frac{1}{2}} + \frac{2}{3}\times(1-e^{0})^{\frac{1}{2}}\right)$ **[1mark]**

$= \left(-\frac{2}{3}\times(1-e^{6})^{\frac{1}{2}}\right) = \boxed{-\frac{2}{3}\sqrt{1-e^{6}}}$ **[2marks]**

2	a) Show that $2x\sin 2x = 4x\sin x\cos x$ b) Hence use integration by parts to prove that $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2x\sin 2x\,dx = \frac{7\pi-3\sqrt{3}}{12}$ **<u>Solution</u>** a) Starting from LHS, $\sin 2x = \sin(x+x)$, Using the expansion of $\sin(A+B)$ $\sin(x+x) = \sin x\cos x + \cos x\sin x$ Therefore $\sin 2x = 2\sin x\cos x$ **[1mark]** **Substitute** $\sin 2x = 2\sin x\cos x$ into $2x\sin 2x$ Hence $2x \times 2\sin x\cos x = 4x\sin x\cos x$ **[1mark]** b) Integration by parts: $\left\{\int v\frac{du}{dx}dx = vu - \int u\frac{dv}{dx}dx\right\}$ $= 2\left[\frac{1}{2}\left(-x\cos 2x - 2\int\frac{1}{2}\cos 2x\,dx\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ **[1mark]** $= 2\left[\frac{1}{2}\left(-x\cos 2x - 2\left(-\frac{1}{4}\sin 2x\right)\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ **[1mark]** $= \left[\left(-x\cos 2x + \frac{1}{2}\sin 2x\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ **[1mark]** $= 2\left(-\frac{\pi}{4}\cos\pi + \frac{1}{2}\sin\pi + \frac{\pi}{12}\cos\frac{\pi}{3} - \frac{1}{2}\sin\frac{\pi}{3}\right)$ **[1mark]** $= 2\left(\frac{7\pi-3\sqrt{3}}{24}\right) = \boxed{\frac{7\pi-3\sqrt{3}}{12}}$ **[1mark]**

3	Prove that $\int_0^{3\pi} e^{2x} \sin 3x \, dx = \frac{3e^{6\pi}+3}{13}$

Solution

Integration by parts formula: $\left\{\int v \frac{du}{dx} dx = vu - \int u \frac{dv}{dx} dx\right\}$

$u = e^{2x}$ Therefore, $\frac{du}{dx} = 2e^{2x}$ **[1mark]**

$\frac{dv}{dx} = \sin 3x$ Therefore, $v = -\frac{1}{3}\cos 3x$ **[1mark]**

Start from LHS, applying integration by parts,

$= -\frac{1}{3}e^{2x}\cos 3x - \int \left(-\frac{2}{3}e^{2x}\cos 3x\right) dx$

$= -\frac{1}{3}e^{2x}\cos 3x - \left(-\frac{2}{3}\int e^{2x}\cos 3x \, dx\right)$ **[1mark]**

Applying integration by parts again,

$= -\frac{1}{3}e^{2x}\cos 3x - \left[-\frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \left(-\frac{2}{3}\int e^{2x}\sin 3x \, dx\right)\right)\right]$

[1mark]

$= -\frac{1}{3}e^{2x}\cos 3x - \left(-\frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x \, dx\right)\right)$

[1mark]

Let, $\int e^{2x}\sin 3x \, dx = u$

$u = -\frac{1}{3}e^{2x}\cos 3x - \left(-\frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}u\right)\right)$

[1mark]

$$\int e^{2x}\sin 3x\,dx = \frac{-3e^{2x}\cos 3x}{13} + \frac{2e^{2x}\sin 3x}{13}$$ **[1mark]**

$$= \left[\frac{-3e^{2x}\cos 3x}{13}\right]_0^{3\pi} + \left[\frac{2e^{2x}\sin 3x}{13}\right]_0^{3\pi}$$ **[1mark]**

Hence, $\int e^{2x}\sin 3x\,dx = \frac{3e^{6\pi}+3}{13}$ **[1mark]**

4	A family of parabolas is defined by differential equation $dy = (5x - 3)dx$ Two members of the family, f(x) and g(x) pass through the points $(4, 5)$ and $(1, 8)$ respectively. a) Find the equations for f(x) and $g(x)$ b) Hence find the area bound between f(x) and $-g(x)$ **<u>Solution</u>** a) $\frac{dy}{dx} = 5x - 3$ **[1mark]** Integrating Both sides $\int \frac{dy}{dx} = 5 \int x\, dx - 3 \int 1\, dx$ $y = \frac{5}{2}x^2 - 3x + c$ **[1mark]** Using $(4, 5)$ $5 = \frac{5}{2}(4^2) - 3(4) + c$ $5 = 5(8) - 3(4) + c$ $c = 5 - 40 + 12$ $\boxed{c = -23}$ **[1mark]** Hence, f$(x) = \frac{5}{2}\, x^2 - 3x - 23$ **[1mark]**

Using $(1, 8)$

$y = \frac{5}{2}x^2 - 3x + c$

$8 = \frac{5}{2}(1^2) - 3(1) + c$

$\boxed{c = \frac{17}{2}}$ **[1mark]**

Hence, $g(x) = \frac{5}{2}x^2 - 3x + \frac{17}{2}$ **[1mark]**

b) If $g(x) = \frac{5}{2}x^2 - 3x + \frac{17}{2}$

Then $-g(x) = -\frac{5}{2}x^2 + 3x - \frac{17}{2}$ **[1mark]**

$\frac{5}{2}\ x^2 - 3x - 23 = -\frac{5}{2}x^2 + 3x - \frac{17}{2}$ **[1mark]**

$5\ x^2 - 6x - 46 = -5\ x^2 + 6x - 17$

$10\ x^2 - 12x - 29 = 0$

$x = 2.41$ (3sf) or $x = -1.21$ (3sf) **[1mark]**

The area bound between the functions will be given as

$$\int_{-1.21}^{2.41} (-g(x)dx - f(x))dx$$

So we have

$\int_{-1.21}^{2.41} \left(-\frac{5}{2}x^2 + 3x - \frac{17}{2}\right) - \int_{-1.21}^{2.41} \left(\frac{5}{2}\ x^2 - 3x - 23\right) dx$

[1mark]

$$= \left[-\frac{5}{6}x^3 + \frac{3}{2}x^2 - \frac{11}{2}x\right]_{-1.21}^{2.41} - \left[\frac{5}{6}x^3 - \frac{3}{2}x^2 - 23x\right]_{-1.21}^{2.41}$$

[2marks]

$= -37.3 - (-76.5) = 39.2$ **[1mark]**

Hence, the area is 39.2 square units

5	a) Express $\frac{3x+4}{x(x+1)}$ in partial fractions b) Hence find the general solution for the differential equation $x(x+1)\frac{dy}{dx} = y(3x+4)$ c) Hence find the particular solution when $x = 2$ and $y = 3$ **<u>Solution</u>** a) $\frac{3x+4}{x(x+1)} = \frac{A}{x} + \frac{B}{(x+1)}$ $3x + 4 = A(x+1) + Bx$ **[1mark]** Let $x = 0$, $\boxed{A = 4}$ **[1mark]** Let $x = -1$, $\boxed{B = -1}$ **[1mark]** Hence, $\frac{3x+4}{x(x+1)} = \frac{4}{x} - \frac{1}{(x+1)}$ **[1mark]** b) $x(x+1)\frac{dy}{dx} = y(3x+4)$ $\frac{1}{y}y' = \frac{3x+4}{x(x+1)}$ Integrating both sides, $\int \frac{1}{y}y'dy = \int \frac{3x+4}{x(x+1)}dx$ **[1mark]** $\ln y = \int \frac{4}{x}dx - \int \frac{1}{x+1}dx$ **[1mark]**

$\ln y = 4\ln x - \ln|x+1| + c$ **[2marks]**

$\boxed{y = \frac{Ax^4}{x+1}}$ where $A = e^c$ **[1mark]**

c)Particular solution when $x = 2$ and $y = 3$

$3 = \frac{A \times 2^4}{2+1}$

$9 = 16A$ therefore, $A = \frac{9}{16}$ **[1mark]**

Hence, the particular solution will be $y = \frac{9x^4}{16(x+1)}$

[1mark]

6	Find the particular solutions for the differential equation

6 Find the particular solutions for the differential equation

$$\operatorname{cosec} x \ \frac{dy}{dx} = e^x \operatorname{cosec} x + 3x$$

when $y = 3$ and $x = \frac{\pi}{3}$

Solution

$\operatorname{cosec} x \ \frac{dy}{dx} = e^x \operatorname{cosec} x + 3x$

$\frac{dy}{dx} = \frac{e^x cosec x}{cosec\ x} + \frac{3x}{cosec\ x}$ **[1mark]**

$\frac{dy}{dx} = e^x + 3x \sin x$ **[1mark]**

Integrating both sides,

$\int dy = \int e^x dx + \int 3x \sin x \, dx$

$y = e^x + 3[-x \cos x - \int(-\cos x)\, dx]$ **[3marks]**

$y = e^x + 3[-x \cos x + \sin x]$

$y = e^x - 3x \cos x + 3 \sin x + c$ **[1mark]**

Substitute in $y = 3$ and $x = \frac{\pi}{3}$

$3 = e^{\frac{\pi}{3}} - 3\left(\frac{\pi}{3}\right) \cos\frac{\pi}{3} + 3 \sin\frac{\pi}{3} + c$ **[1mark]**

$c = 3 - 2.85 - \frac{3\sqrt{3}}{2} + \frac{\pi}{2}$

$c = -0.877(3\text{sf})$ **[1mark]**

$\boxed{y = e^x - 3x \cos x + 3 \sin x - 0.877}$