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Year 2 (A-Level)

Integration Set 1A

Reverse chain rule, e^x, Partial fractions, $\frac{f'(x)}{f(x)}$ and Trig

(Solutions)

The marks shown are for guidance purposes only

The questions have been repeated here for your convenience

1

a) Find $\int \frac{e^{5x}}{2e^{5x}+1}dx$

b) Hence evaluate $\int_0^1 \frac{e^{5x}}{2e^{5x}+1}dx$

Solution

a) Let $u = 2e^{5x} + 1$

$\frac{du}{dx} = 10e^{5x}$ hence $dx = \frac{du}{10e^{5x}}$ **[1mark]**

Therefore, $\int \frac{e^{5x}}{2e^{5x}+1}dx = \int \frac{e^{5x}}{u} \times \frac{du}{10e^{5x}}$

So we have $\int \frac{\cancel{e^{5x}}}{u} \times \frac{du}{10\cancel{e^{5x}}} = \int \frac{1}{10u}du$ **[1mark]**

$\int \frac{1}{10u}du = \frac{1}{10}\ln u + C$ **[1mark]**

Hence, $\int \frac{e^{5x}}{2e^{5x}+1}dx = \frac{1}{10}\ln\left|2e^{5x} + 1\right| + C$ **[1mark]**

b) Substitute the limits into the integral from **part a**

$\int_0^1 \frac{e^{5x}}{2e^{5x}+1}dx = \left[\frac{1}{10}\ln\left|2e^{5x} + 1\right|\right]_0^1$ **[1mark]**

$$\int_0^1 \frac{e^{5x}}{2e^{5x} + 1}dx = \left(\frac{1}{10}\ln\left|2e^{5(1)} + 1\right|\right) - \left(\frac{1}{10}\ln\left|2e^{5(0)} + 1\right|\right)$$

$= 0.460(3\text{sf})$

[2marks]

2	A function is defined by the equation $\mathrm{f}(x) = \sin 3x$

a) Sketch the graph of $\mathrm{f}(x)$ for $0 \le x \le 2\pi$

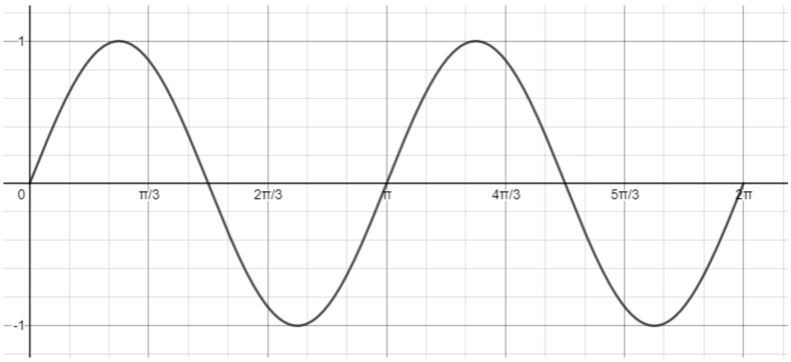
b) Show that $\sin 3x = 3\sin x\left(\cos^2 x - \frac{1}{3}\sin^2 x\right)$

c) Hence evaluate $\int_0^{\pi} (\sin 3x)dx$

Give your answer as an exact value

Solution

a) **[1mark]**



b) Start from the LHS $\sin(3x) = \sin(2x + x)$

Now use the compound angle formula to expand

$\sin(2x + x) = \sin 2x\cos x + \cos 2x\sin x$ **[1mark]**

Note that

$\sin 2x = 2\sin x\cos x$ **[1mark]**

$\cos 2x = \cos^2 x - \sin^2 x$ **[1mark]**

Substitute these into the expression

$\sin 2x\cos x + \cos 2x\sin x$

Therefore, $\sin 2x\cos x + \cos 2x\sin x$

$= 2\cos x(\sin x\cos x) + \sin x(\cos^2 x - \sin^2 x)$ **[1mark]**

$2\sin x\cos^2 x + \sin x\cos^2 x - \sin^3 x$

$= 3\sin x\cos^2 x - \sin^3 x$ **[1mark]**

$= 3\sin x\left(\cos^2 x - \frac{1}{3}\sin^2 x\right)$ **[1mark]**

c) Let $u = 3x$

Then, $du = 3dx$

$\int_0^{3\pi} (\sin u) \frac{1}{3} du = \frac{1}{3} \int_0^{3\pi} (\sin u)\, du$

$= \frac{1}{3} [-\cos u]_0^{3\pi}$ **[1mark]**

$= \frac{1}{3} [-\cos 3\pi - (-\cos 0)] = \frac{1}{3} [1 + 1]$ **[1mark]**

Hence, $\boxed{\int_0^{\pi} (\sin 3x) dx = \frac{2}{3}}$ **[1mark]**

3	A function f(x) is defined as $y = \mathrm{f}(x)$ The gradient function of $\mathrm{f}'(x) = 5x(kx + 7x^2)$ where k is a constant The equation of a tangent to $\mathrm{f}(x)$ when $x = -1$ is $y = 10x + 3$ a)Find the value of k b) Find f(x) c)Hence evaluate $\int_{-1\frac{1}{4}}^{0} \mathrm{f}(x)\, dx$ **<u>Solution</u>** a)$\mathrm{f}'(x) = y'$ Therefore, $\mathrm{f}'(x) = 10$ at $x = -1$ The curve and the tangent have equal gradients at $x = -1$ Therefore, $5x(kx + 7x^2) = 10$ when $x = -1$ $5(-1)(-k + 7) = 10$ $5k - 35 = 10$ Therefore, $\boxed{k = 9}$ **[2marks]**

b) $f'(x) = 5x(9x + 7x^2) = 45x^2 + 35x^3$ **[1mark]**

$\int f'(x)dx = \int 45x^2dx + \int 35x^3dx$

$f(x) = \frac{45}{3}x^3 + \frac{35}{4}x^4 + c$ **[1mark]**

$10x + 3 = \frac{45}{3}x^3 + \frac{35}{4}x^4 + c$

At $x = -1$,

$10(-1) + 3 = -15 + \frac{35}{4} + c$ **[1mark]**

$c = -\frac{3}{4}$ **[1mark]**

$\boxed{f(x) = 15x^3 + \frac{35}{4}x^4 - \frac{3}{4}}$

c) $\int 15x^3dx + \int \frac{35}{4}x^4dx - \int \frac{3}{4}dx$

$= \left[\frac{15}{4}x^4\right]_{-1\frac{1}{4}}^{0} + \left[\frac{35}{20}x^5\right]_{-1\frac{1}{4}}^{0} - \left[\frac{3}{4}x\right]_{-1\frac{1}{4}}^{0}$ **[3marks]**

$= -\frac{9375}{1024} + \frac{21875}{4096} - \frac{15}{16} = \boxed{-\frac{19465}{4096}}$ **[1mark]**

4 Given that $\frac{2x+5}{2x^2-5x+3} = \frac{A}{\mathrm{f}(x)} + \frac{B}{g(x)}$

a)Find the values of A and B

b) Hence find $\int \frac{2x+5}{2x^2-5x+3}\,dx$

Solution

a)Factorise the denominator and write the expression as partial fractions.

$\frac{2x+5}{2x^2-5x+3} = \frac{2x+5}{2x^2-2x-3x+3} = \frac{2x+5}{(2x-3)(x-1)}$ **[1mark]**

$\frac{2x+5}{(2x-3)(x-1)} = \frac{A}{(2x-3)} + \frac{B}{(x-1)}$

Solving as partial fractions

$\frac{(2x+5)\cancel{(2x-3)(x-1)}}{\cancel{(2x-3)(x-1)}} = \frac{A\cancel{(2x-3)}(x-1)}{\cancel{(2x-3)}} + \frac{B(2x-3)\cancel{(x-1)}}{\cancel{(x-1)}}$

$2x + 5 = A(x-1) + B(2x-3)$ **[1mark]**

Let $x = 1$

$2 + 5 = B(-1)$

$\boxed{B = -7}$ **[1mark]**

Let $x = \frac{3}{2}$

$8 = A\left(\frac{1}{2}\right)$

$\boxed{A = 16}$ **[1mark]**

Hence, $\frac{2x+5}{(2x-3)(x-1)} = \frac{16}{(2x-3)} - \frac{7}{(x-1)}$

b) $\int \frac{2x+5}{2x^2-5x+3} dx = \int \frac{16}{2x-3} dx + \int \frac{-7}{x-1} dx$

Apply *u*-substitution,

$u = 2x - 3$

$du = 2dx$

$\int \frac{2x+5}{2x^2-5x+3} dx = \int \frac{16}{2u} du - 7\ln|x-1| + c$

$$\boxed{\int \frac{2x+5}{2x^2-5x+3} dx = 8\ln|2x-3| - 7\ln|x-1| + c}$$

[2marks]

5 Given that $f(x) = \frac{x^2}{(x-2)(x+3)^2}$

a) Express $f(x)$ in partial fractions

b) Find $\int f(x)\, dx$

c) Hence evaluate $\int_{-\frac{5}{2}}^{0} f(x) dx$

Solution

a) $\frac{x^2}{(x-2)(x+3)^2} = \frac{A}{(x-2)} + \frac{B}{(x+3)} + \frac{C}{(x+3)^2}$ **[1mark]**

$$\frac{x^2(x-2)(x+3)^2}{(x-2)(x+3)^2} = \frac{A(x-2)(x+3)^2}{(x-2)} + \frac{B(x-2)(x+3)^2}{(x+3)} + \frac{C(x-2)(x+3)^2}{(x+3)^2}$$

$x^2 = A(x+3)^2 + B(x+3)(x-2) + C(x-2)$ **[1mark]**

Let $x = 2$

$4 = A(5)^2$

$\boxed{A = \frac{4}{25}}$ **[1mark]**

Let $x = -3$

$9 = -5C$

$\boxed{C = -\frac{9}{5}}$ **[1mark]**

$x^2 = \frac{4}{25}(x+3)^2 + B(x+3)(x-2) - \frac{9}{5}(x-2)$

Comparing co-efficient,

$1 = \frac{4}{25} + B$ therefore, $\boxed{B = \frac{21}{25}}$ **[1mark]**

Hence, $\boxed{\frac{x^2}{(x-2)(x+3)^2} = \frac{4}{25(x-2)} + \frac{21}{25(x+3)} - \frac{9}{5(x+3)^2}}$ **[1mark]**

b) $\int f(x)dx = \int \frac{4}{25(x-2)}dx + \int \frac{21}{25(x+3)}dx - \int \frac{9}{5(x+3)^2}dx$

$= \frac{4}{25}\ln|x-2| + \frac{21}{25}\ln|x+3| - \int \frac{9}{5(x+3)^2}dx$

Apply u-substitution,

$u = x + 3$ therefore, $du = dx$

$\int f(x)dx = \frac{4}{25}\ln|x-2| + \frac{21}{25}\ln|x+3| - \int \frac{9}{5(u)^2}du$

$= \frac{4}{25}\ln|x-2| + \frac{21}{25}\ln|x+3| + \frac{9}{5}u^{-1}$

$\boxed{\int f(x)dx = \frac{4}{25}\ln|x-2| + \frac{21}{25}\ln|x+3| + \frac{9}{5(x+3)} + C}$ **[4marks]**

c) $\int_{-\frac{5}{2}}^{0} f(x)dx = \left[\frac{4}{25}\ln|x-2| + \frac{21}{25}\ln|x+3| + \frac{9}{5(x+3)}\right]_{-\frac{5}{2}}^{0}$

$= -1.62$ (3sf) **[2marks]**

6	a) Show that $\cos 3x$ can be written as $\cos x(1-4\sin^2 x)$ b) Hence solve $\cos 3x = \cos x$ for $0 \le x \le 2\pi$ c) Evaluate $\int_0^{\frac{1}{6}\pi} \cos 3x\, dx$ Give your answer as an exact value **<u>Solution</u>** a) From LHS: $\cos 3x = \cos(2x + x)$ $\cos(2x + x) = \cos 2x \cos x - sin2xsinx$ **[1mark]** Note that $\cos 2x = 1 - 2\sin^2 x$ and $\sin 2x = 2\sin x \cos x$ Therefore, $\cos 2x \cos x - \sin 2x \sin x$ $= (1-2\sin^2 x)\cos x - (2\sin x\cos x)\sin x$ **[1mark]** $= \cos x - 2\sin^2 x\cos x - 2\sin^2 x\cos x$ **[1mark]** $= \cos x - 4\sin^2 x\cos x$ $= \cos x(1-4\sin^2 x)$ **[1mark]** Hence, $\cos 3x = \cos x(1-4\sin^2 x)$

b) $\cos 3x = \cos x$

$\cos 3x - \cos x = 0$

$\cos(2x + x) - \cos x = 0$

Using,

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\cos 2x \cos x - \sin 2x \sin x - \cos x = 0$ **[1mark]**

$\cos 2x \cos x - 2\sin x \cos x \sin x - \cos x = 0$ **[1mark]**

$\cos 2x \cos x - 2\sin^2 x \cos x - \cos x = 0$

$(1 - 2\sin^2 x)\cos x - 2\sin^2 x \cos x - \cos x = 0$

$\cancel{\cos x} - 2\sin^2 x \cos x - 2\sin^2 x \cos x - \cancel{\cos x} = 0$

$-4\sin^2 x \cos x = 0$ **[1mark]**

$\sin^2 x = 0$ or $\cos x = 0$ **[1mark]**

Therefore, $\boxed{x = 0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}}$ **[1mark]**

c) Let $u = 3x$ therefore, $du = 3dx$

$\int_0^{\frac{1}{6}\pi} \cos 3x \, dx = \int_0^{\frac{\pi}{2}} \cos u \frac{du}{3}$ **[2marks]**

$= \frac{1}{3}[\sin u]_0^{\frac{\pi}{2}} = \frac{1}{3}(1 - 0) = \frac{1}{3}$ **[1mark]**

Hence $\boxed{\int_0^{\frac{1}{6}\pi} \cos 3x \, dx = \frac{1}{3}}$