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Year 2 (A-Level)

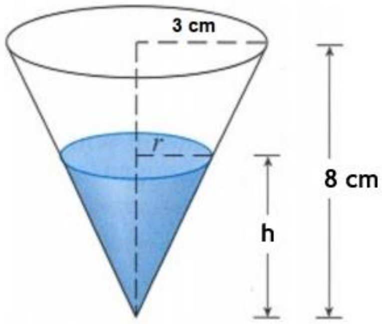
Differentiation Set 1B
(Product and Quotient Rules, Implicit differentiation)
(Solutions)

The marks shown are for guidance purposes only

The questions are repeated here for your convenience

1	Given that $y = \frac{3\sin x + 2\sin 2x}{2 + 3\cos x + 2\cos 2x}$ a) Show that $y = \tan x$ b) Hence find $\frac{dy}{dx}$ **<u>Solution</u>** a) Use the identities $\sin 2x = 2\sin x \cos x$ and $\cos 2x = 2\cos^2 x - 1$ $y = \frac{3\sin x + 2\,(2\sin x \cos x)}{2 + 3\cos x + 2(2\cos^2 x - 1)}$ **[2marks]** $= \frac{3\sin x + 4\sin x \cos x}{2 + 3\cos x + 4\cos^2 x - 2}$ $= \frac{\sin x \cancel{(3 + 4\cos x)}}{\cos x \cancel{(3 + 4\cos x)}} = \frac{\sin x}{\cos x}$ **[1mark]** Therefore, $\boxed{y = \tan x}$ **[1mark]** b) Standard derivative: if $y = \tan x$ then $\frac{dy}{dx} = \sec^2 x$ **[1mark]** {Alternatively, by writing $\tan x = \frac{\sin x}{\cos x}$, you may apply the quotient rule}

2	The angle of a sector is 2.168 rad. The radius of the sector is increasing at a rate of 0.95 ms^{-1} At what rate is the area increasing when the radius of the sector is 55 cm? **<u>Solution</u>** The area of a sector $= \frac{1}{2}r^2\theta$ where θ in radians Differentiating the above equation $\frac{dA}{dt} = \frac{1}{\cancel{2}} \times \cancel{2}r \times \frac{dr}{dt} \times \theta$ **[1mark]** $= r \times \frac{dr}{dt} \times \theta$ $\frac{dA}{dt} = 0.55 \times 0.95 \times 2.168$ **[1mark]** $\boxed{\frac{dA}{dt} = 1.13\ \text{m}^2\text{s}^{-1}(3\text{sf})}$ **[1mark]**

3	Water is being poured into a hollow cone at a rate of 5 cm^3min^{-1}  The height and radius of the cone are 8 cm and 3 cm respectively. Calculate the rate at which the water level is rising when the depth of water is 5.5 cm. **<u>Solution</u>** The volume of a cone is given by: $V = \frac{1}{3}\pi r^2 h$ By using similar triangles, we find that, $\frac{8}{h} = \frac{3}{r}$ Therefore, $r = \frac{3}{8}h$ Substitute $r = \frac{3}{8}h$ into $V = \frac{1}{3}\pi r^2 h$ **[1mark]** Therefore, $\mathrm{V} = \frac{1}{3}\pi\left(\frac{3}{8}h\right)^2 h$ $V = \frac{3}{64}\pi h^3$ **[1mark]**

Differentiate the above equation

$$\frac{dV}{dh} = \frac{d}{dh}\left(\frac{3}{64}\pi h^3\right)$$

$$\frac{dV}{dh} = \frac{9}{64}\pi h^2$$ **[1mark]**

$$dV = \frac{9}{64}\pi h^2 dh$$

$$dh = \frac{64dV}{9\pi h^2}$$ **[1mark]**

$$dh = \frac{64\times 5}{9\pi(5.5)^2} = 0.374$$ **[1mark]**

$\boxed{dh = 0.374}$ cm/m (3sf)

4 Given that $y = \frac{\tan 2x}{x^5}$

a) Find $\frac{dy}{dx}$

b) Hence evaluate $\frac{dy}{dx}$ at $x = \frac{\pi}{3}$

Solution

a) Apply quotient rule as follows:

$$\frac{dy}{dx} = \frac{\left(\frac{d}{dx}(\tan 2x)\right)x^5 - \left(\frac{d}{dx}(x^5)\right)\tan 2x}{(x^5)^2}$$ **[1mark]**

$$\frac{dy}{dx} = \frac{(2\sec^2 2x)x^5 - (5x^4)\tan 2x}{x^{10}}$$ **[1mark]**

$$= \frac{2x^5 \sec^2 2x - 5x^4 \tan 2x}{x^{10}}$$ **[1mark]**

$$\frac{dy}{dx} = \frac{\cancel{x^4}(2x\sec^2 2x - 5\tan 2x)}{x^{\cancel{10}}}$$ **[1mark]**

$$= \frac{2x\sec^2 2x - 5\tan 2x}{x^6}$$ **[1mark]**

$$\boxed{\frac{dy}{dx} = \frac{2x\sec^2 2x - 5\tan 2x}{x^6}}$$

b)

$$\left[\frac{dy}{dx}\right]_{\frac{\pi}{3}} = \frac{2\left(\frac{\pi}{3}\right)\sec^2 2\left(\frac{\pi}{3}\right) - 5\tan 2\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)^6}$$

$$\left[\frac{dy}{dx}\right]_{\frac{\pi}{3}} = \frac{\frac{2\pi}{3}\sec^2\left(\frac{2\pi}{3}\right) - 5\tan\left(\frac{2\pi}{3}\right)}{\left(\frac{\pi}{3}\right)^6}$$ **[1mark]**

$$= \frac{\left(\frac{2\pi}{3}\times 4\right) - \left(5\times(-\sqrt{3})\right)}{\left(\frac{\pi}{3}\right)^6}$$ **[1mark]**

$$\left[\frac{dy}{dx}\right]_{\frac{\pi}{3}} = \frac{\frac{8\pi}{3} + 5\sqrt{3}}{\frac{\pi^6}{729}} = 12.9$$ **[1mark]**

$$\boxed{\left[\frac{dy}{dx}\right]_{\frac{\pi}{3}} = 12.9}\text{(3sf)}$$

5	The equation of a curve is given as $y = x^2 \ln x$

a)Find the roots of the curve

b) Find $\frac{dy}{dx}$ and hence the exact coordinates of the stationary point of the curve

c)Hence show that $\frac{d^2y}{dx^2} = 2\ln x + 3$

<u>Solution</u>

a)The roots of the curve are

$0 = x^2 \ln x$ **[1mark]**

So, $\ln x = 0$ or $x = 0$

If $0 = \ln x$

Then, $x = e^0 = 1$

Hence the roots are $x = 0$ or $x = 1$ **[1mark]**

Verify Solutions:

$0 = 0^2 \ln 0$ is undefined therefore false

$0 = 1^2 \ln 1$ is true

Hence, the only solution is, $\boxed{x = 1}$ **[1mark]**

b)

Apply the product rule

$\frac{dy}{dx} = \frac{d}{dx}(x^2)\ln x + \frac{d}{dx}(\ln x)x^2$ **[1mark]**

$\frac{dy}{dx} = 2x\ln x + \frac{1}{x}x^2 = 2x\ln x + x$ **[3marks]**

Therefore, $\boxed{\frac{dy}{dx} = 2x\ln x + x}$

Stationary point occurs when $\frac{dy}{dx} = 0$

Therefore, $\frac{dy}{dx} = 2x \ln x + x = 0$ **[1mark]**

$x(2 \ln x + 1) = 0$

Therefore, either $2 \ln x + 1 = 0$ or $x = 0$

$\ln x = -\frac{1}{2}$ **[1mark]**

$x = e^{-\frac{1}{2}}$

Verify the solution:

When $x = 0$, the expression will be undefined

Therefore, $x = e^{-\frac{1}{2}}$ is the only solution

[1mark]

Now find the y-coordinate of the stationary point,

$y = \left(e^{-\frac{1}{2}}\right)^2 \ln e^{-\frac{1}{2}}$

$= -\frac{1}{2e}$ **[1mark]**

Hence, the coordinate of the stationary point is

$\left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)$

c)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x\ln x + x)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x\ln x) + \frac{d}{dx}(x)$$

Apply the product rule to the first part:

$$\frac{d^2y}{dx^2} = 2\left(\frac{d}{dx}(x)\ln x + \frac{d}{dx}(\ln x)x\right) + 1$$ **[1mark]**

$$\frac{d^2y}{dx^2} = 2\left(1\ln x + \frac{1}{x}x\right) + 1$$ **[3marks]**

$$= 2\ln x + 3$$ **[1mark]**

Hence, $\boxed{\frac{d^2y}{dx^2} = 2\ln x + 3}$

6	A curve is defined implicitly by the equation $x^2 + kxy - 3xy^2 = 5$ Where k is a constant. The gradient of the curve at the point $(-1, 2)$ is $\frac{3}{4}$ Find the value of k

Solution

a)Use implicit differentiation

$x^2 + kxy - 3xy^2 = 5$

$\frac{d}{dx}(x^2 + kxy - 3xy^2) = \frac{d}{dx}(5)$ **[1mark]**

$2x + kx\frac{dy}{dx} + ky - 3\left(y^2 + 2xy\frac{dy}{dx}\right) = 0$ **[4marks]**

$2x + kx\frac{dy}{dx} + ky - 3y^2 - 6xy\frac{dy}{dx} = 0$ **[1mark]**

$kx\frac{dy}{dx} - 6xy\frac{dy}{dx} = -2x - ky + 3y^2$

$\frac{dy}{dx} = \frac{-2x-ky+3y^2}{kx-6xy}$ **[1mark]**

$\frac{dy}{dx} = \frac{3}{4}$ at $(-1,2)$

Therefore, $\frac{3}{4} = \frac{-2(-1)-k(2)+3(2)^2}{k(-1)-6(-1)(2)}$ **[1mark]**

$\frac{3}{4} = \frac{2-2k+12}{-k+12}$

$-3k + 36 = 8 - 8k + 48$ **[1mark]**

$5k = 20$ therefore, $k = 4$ **[1mark]**