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Year 2 (A-Level)

Binomial Expansion – Solutions

The marks shown are for guidance purposes only

The questions are repeated here for your convenience

1

a)Express $\sqrt{15}$ in the form $k(1-x)^{\frac{1}{2}}$ where k is a constant

b)If x^5 and higher powers can be ignored, find an approximation for $5\sqrt{15}$

c)Calculate the percentage error in your approximation

Solution

a)

$$15 = 25 - 10$$

Therefore, $(15)^{\frac{1}{2}} = (25-10)^{\frac{1}{2}}$

$$= (25)^{\frac{1}{2}}\left(1-\frac{10}{25}\right)^{\frac{1}{2}}$$ **[2marks]**

Hence, $\boxed{\sqrt{15} = 5\left(1-\frac{2}{5}\right)^{\frac{1}{2}}}$ **[1mark]**

Where $k = 5$ and $x = \frac{2}{5}$

b) $5\sqrt{15} = 25\left(1-\frac{2}{5}\right)^{\frac{1}{2}}$,

we need to expand this expression up to and including the term in x^4

$$\left(1-\frac{2}{5}\right)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)\left(-\frac{2}{5}\right)$$
$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(-\frac{2}{5}\right)^2$$
$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(-\frac{2}{5}\right)^3$$
$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{7}{2}\right)}{24}\left(-\frac{2}{5}\right)^4$$

[4marks]

$= 1 - 0.2 - 0.02 - 0.004 - 0.0014 = 0.7746$ **[1mark]**

$5\sqrt{15} = 25(0.7746)$

Therefore, $\boxed{5\sqrt{15} \approx 19.365}$ **[1mark]**

c)From the expansion, $5\sqrt{15} = 19.365$

From the calculator, $5\sqrt{15} = 19.36491673$

Error$= 19.36491673 - 19.365 = 0.0000833$(3sf) **[1mark]**

%Error$= \frac{0.0000833}{19.36491673} \times 100 = 0.00043\%$ **[1mark]**

2	The coefficient of the third term in the expansion of the expression $(1+5x)^{\frac{1}{3}}$ is equal to the coefficient of the fourth term in the expansion of the expression $(1+px)^5$ Find the value of *p*. Give your answer to 3 significant figures **<u>Solution</u>** Find the third term from $(1+5x)^{\frac{1}{3}}$ $(1+5x)^{\frac{1}{3}} = 1 + \left(\frac{1}{3}\right)(5x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}(5x)^2 + \cdots$ **[1mark]** $(1+5x)^{\frac{1}{3}} = 1 + \frac{5}{3}x - \frac{25}{9}x^2$ **[1mark]** Therefore, the coefficient of the third term is $-\frac{25}{9}$ Find the fourth term from $(1+px)^5$ $\binom{5}{3}(1)^2(px)^3 = 10p^3x^3$ **[1mark]** The coefficient of the fourth term is $10p^3$ Therefore, $-\frac{25}{9} = 10p^3$ **[1mark]** $-\frac{5}{18} = p^3$ $p = \sqrt[3]{-\frac{5}{18}}$ Therefore, $\boxed{p = -0.653}$ (2sf) **[1mark]**

3	Expand the expression $\frac{1}{1+2x}$ up to and including the term in x^4 and use your expansion to find an approximate value for $\frac{1}{1.0125}$ **<u>Solution</u>** $\frac{1}{1+2x} = 1 + (-1)(2x)$ $+ \frac{(-1)(-2)}{2}(2x)^2$ $+ \frac{(-1)(-2)(-3)}{6}(2x)^3$ $+ \frac{(-1)(-2)(-3)(-4)}{24}(2x)^4$ $= 1 - 2x + 4x^2 - 8x^3 + 16x^4$ **[2marks]** Comparing $\frac{1}{1+2x}$ and $\frac{1}{1.0125}$ $2x = 0.0125$ therefore, $x = 0.00625$ **[1mark]** Substitute $x = 0.00625$ into the expansion $= 1 - 2(0.00625) + 4(0.00625)^2 - 8(0.00625)^3 + 16(0.00625)^4$ $= 0.9876543254$ **[1mark]** Hence, $\frac{1}{1.0125} \approx 0.988$ (3sf) **[1mark]**

4	a)Obtain the first 5 terms in the expansion of the expression $(1-16x)^{\frac{1}{4}}$ b) State the validity of your expansion c)By using the first 3 terms and substituting $x=\frac{1}{625}$ find the approximate value of $\sqrt[4]{609}$ **Solution** a)Find the first 5 terms as follows: $(1-16x)^{\frac{1}{4}}$ $=1+\left(\frac{1}{4}\right)(-16x)$ $+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2}(-16x)^2$ $+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{6}(-16x)^3$ $+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\left(-\frac{11}{4}\right)}{24}(-16x)^4$ $(1-16x)^{\frac{1}{4}}=1-4x-24x^2-224x^3-2464x^4$ **[3marks]** b) The expansion is valid when $\lvert 16x\rvert<1$ Therefore, $\lvert x\rvert<\frac{1}{16}$ **[1mark]**

c)By using $(1-16x)^{\frac{1}{4}} = 1-4x-24x^2$

Substitute $x = \frac{1}{625}$ into the above equation

$$\left(1-\frac{16}{625}\right)^{\frac{1}{4}} = 1-4\left(\frac{1}{625}\right)-24\left(\frac{1}{625}\right)^2$$ **[1mark]**

$$\left(\frac{625}{625}-\frac{16}{625}\right)^{\frac{1}{4}} = 1-4\left(\frac{1}{625}\right)-24\left(\frac{1}{625}\right)^2$$

$$\left(\frac{609}{625}\right)^{\frac{1}{4}} = 1-4\left(\frac{1}{625}\right)-24\left(\frac{1}{625}\right)^2$$ **[1mark]**

$$\frac{\sqrt[4]{609}}{\sqrt[4]{625}} = 0.99353856$$ **[1mark]**

So we have, $\frac{\sqrt[4]{609}}{5} = 0.99353856$

Therefore, $\sqrt[4]{609} = 5 \times 0.99353856 = 4.97$ (3sf)
[1mark]

5 Given that $f(x) = \frac{x-3}{x^2+5x-6}$

a) Express $f(x)$ as partial fractions

b) Hence expand $f(x)$ up to and including the term in x^3

c) State the validity of the expansion

d) Find the equation of a tangent to $f(x)$ at $x = -2$

Give your answer in the form $Ax + By + C = 0$

Solution

a) $\frac{x-3}{x^2+5x-6} = \frac{(x-3)}{(x+6)(x-1)}$

$\frac{(x-3)}{(x+6)(x-1)} = \frac{A}{(x+6)} + \frac{B}{(x-1)}$ **[1mark]**

$(x-3) = A(x-1) + B(x+6)$ **[1mark]**

When $x = 1$

$(1-3) = A(1-1) + B(1+6)$

$-2 = 7B$

$B = -\frac{2}{7}$ **[1mark]**

When $x = -6$

$(-6-3) = A(-6-1) + B(-6+6)$

$-9 = -7A$

$A = \frac{7}{9}$ **[1mark]**

Therefore, $\boxed{f(x) = \frac{9}{7(x+6)} - \frac{2}{7(x-1)}}$ **[1mark]**

b)Expanding f(x):

$$\frac{9}{7(x+6)} - \frac{2}{7(x-1)} = \frac{9}{7}(x+6)^{-1} - \frac{2}{7}(x-1)^{-1}$$ **[1mark]**

$$= \left(\frac{9}{7}\right)(6^{-1})\left(1+\frac{x}{6}\right)^{-1} + \left(-\frac{2}{7}\right)(-1^{-1})(1+x)^{-1}$$

$$= \frac{3}{14}\left[1 - \frac{1}{6}x + \frac{1}{36}x^2 - \frac{1}{216}x^3\right] + \frac{2}{7}[1 + x + x^2 + x^3]$$

[4marks]

$$\frac{9}{7(x+6)} - \frac{2}{7(x-1)}$$

$$= \left[\frac{3}{14} - \frac{1}{28}x + \frac{1}{168}x^2 - \frac{1}{1008}x^3\right]$$

$$+ \left[\frac{2}{7} + \frac{2}{7}x + \frac{2}{7}x^2 + \frac{2}{7}x^3\right]$$ **[2Marks]**

Hence, $\boxed{\frac{9}{7(x+6)} - \frac{2}{7(x-1)} = \frac{1}{2} + \frac{1}{4}x + \frac{7}{24}x^2 + \frac{41}{144}x^3}$

[2marks]

c)$|x| < 1$ and $-6 < x < 6$

Hence, the range of validity will be $-1 < x < 1$

[2marks]

d)

$y = f'(x) = \frac{d}{dx}\left[\frac{1}{2} + \frac{1}{4}x + \frac{7}{24}x^2 + \frac{41}{144}x^3\right]$

$f'(x) = \frac{1}{4} + \frac{7}{12}x + \frac{41}{48}x^2$ **[2marks]**

At $x = -2$

$f'(-2) = \frac{1}{4} + \frac{7}{12}(-2) + \frac{41}{48}(-2)^2 = \frac{5}{2}$ **[1mark]**

The tangent will have the equation, $y = \frac{5}{2}x + c$

When $x = -2$

$f(-2) = \frac{9}{7(-2+6)} - \frac{2}{7(-2-1)} = \frac{5}{12}$ **[1Mark]**

Substitute $\left(-2, \frac{5}{12}\right)$ into $y = \frac{5}{2}x + c$

$\frac{5}{12} = \frac{5}{2} \times -2 + c$

$\frac{5}{12} = -5 + c$

$c = \frac{65}{12}$ **[1mark]**

Hence, the equation of the tangent is $y = \frac{5}{2}x + \frac{65}{12}$

$12y = 60x + 65$

$-30x + 12y - 65 = 0$ **[1mark]**

6	Given that $f(x) = \frac{x+1}{(x-2)(x+1)^2}$ a)Expression $f(x)$ as partial fractions b)If x is small so that x^4 and higher powers are ignored, express $f(x)$ in the form $Ax^3 + Bx^2 + Cx + D$ **<u>Solution</u>** a) $\frac{x+1}{(x-2)(x+1)^2} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$ **[1mark]** $x + 1 = A(x + 1)^2 + B(x - 2)(x + 1) + C(x - 2)$ **[1mark]** When $x = -1$ then $C = 0$ when $x = 2$ then $A = \frac{1}{3}$ **[2marks]** Comparing coefficients of x^2 $0 = A + B$ So, $B = -A$ Therefore, $B = -\frac{1}{3}$ **[1mark]** Hence, $f(x) = \frac{1}{3(x-2)} - \frac{1}{3(x+1)}$ **[1mark]**

b) $f(x) = \frac{1}{3}[(x-2)^{-1} - (x+1)^{-1}]$ **[1mark]**

$= \frac{1}{3}[-\frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8}\right) - (1 - x + x^2 - x^3)]$

[3marks]

$= \frac{1}{3}[-\frac{3}{2} + \frac{3}{4}x - \frac{9}{8}x^2 + \frac{15}{16}x^3]$ **[1mark]**

$f(x) = \frac{5}{16}x^3 - \frac{3}{8}x^2 + \frac{1}{4}x - \frac{1}{2}$ **[1mark]**